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Stability of rotating shear flows in shallow water

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Asymptotic methods are used to determine the dispersion equation for disturbances of rotating parallel flows in shallow water. From this equation the unstable modes and their growth rates are determined. The solution involves seven asymptotic expansions which are matched together. The results supplement and extend those which have been obtained previously using numerical methods by Griffiths, Killworth & Stern (1982) and by Hayashi & Young (1987).

1. Introduction

The stability of a zonal flow in a shallow layer of fluid on a rotating sphere was studied by Griffiths, Killworth & Stern (1982) using the f-plane approximation and by Hayashi & Young (1987) using the equatorial β-plane approximation. In both cases the resulting eigenvalue problem was solved numerically. We shall solve this problem analytically by asymptotic methods in the f-plane approximation. Our results will confirm and supplement the previous ones, and our procedure will show how to solve similar problems. For example it should be applicable to the problem of instability of a shear flow in shallow water, which was treated numerically by Satomura (1981*a*, *b*).

The analysis shows that instability occurs only when two different modes resonate, i.e. have the same wavenumber and same phase velocity, as Hayashi & Young (1987) found. There can be two modes only if the governing ordinary differential equation has two turning points. This provides a very simple necessary condition for instability – a certain coefficient in the equation must vanish at two points in the interval. In the case of the equation (2.2), which we solve, this condition requires $c^2 < \frac{3}{8}$ for instability, where c is the scaled phase velocity. Our analysis leads to the sharper condition that $c^2 < \frac{1}{4}$ is necessary for instability. Of course only for certain particular values of c satisfying this condition, and for particular wavenumbers, does instability occur. These unstable values and the associated growth rates will be determined.

2. Formulation

Let us consider the flow of shallow water on the surface of a rotating sphere in the f-plane approximation. There is a steady zonal flow with zero potential vorticity, which can be written in dimensionless variables as

$$U(Y) = Y - \tfrac{1}{2}, \quad H(Y) = \tfrac{1}{2}Y(1-Y), \quad 0 \leqslant Y \leqslant 1. \tag{2.1}$$

Here U is the velocity in the x- or eastward direction, H is the depth of the fluid, and the velocity in the Y-direction is zero. We consider a perturbation of this flow in which the x-component of velocity is $U(Y)+e^{ik(x-ct)}W(Y)$, with corresponding changes in the Y-component and in the depth. Then W satisfies the following linear equation (Griffiths *et al.* 1982, eq. (3.4); Hayashi & Young 1987, eq. (3.10)):

$$\frac{d}{dY}\left[H\frac{dW}{dY}\right]-k^2[H-(U-c)^2]W=0, \quad 0<Y<1.$$

When (2.1) is used in this equation it becomes

$$\frac{d}{dY}\left[\tfrac{1}{2}Y(1-Y)\frac{dW}{dY}\right]+k^2[\tfrac{1}{2}Y(Y-1)+(Y-\tfrac{1}{2}-c)^2]W=0, \quad 0<Y<1. \tag{2.2}$$

At the edges of the current, where $H(Y)=0$, $W(Y)$ must be finite so we have

$$W(0), W(1) \text{ finite}. \tag{2.3}$$

Our goal is to determine the eigenvalues c and the eigenfunctions W of (2.2) and (2.3). We note that the equation is unchanged under the transformation $(c, Y)\rightarrow(-c, 1-Y)$. Therefore it suffices to consider $c \geqslant 0$.

3. Asymptotic solution for $c^2 > \frac{3}{8}$

To solve (2.2) we shall seek an asymptotic expansion of $W(Y,k)$ for $k \gg 1$. The nature of the expansion depends upon the sign of the coefficient of k^2. We see that this coefficient is positive for all Y in the interval if $c^2 > \frac{3}{8}$. However if $c^2 < \frac{3}{8}$ the coefficient vanishes at the two points

$$Y_{\pm}=\frac{1}{2}+\frac{2c}{3}\pm\frac{1}{6}(3-8c^2)^{\frac{1}{2}}. \tag{3.1}$$

It is negative between these points and positive outside them.

These considerations show that for $c^2 > \frac{3}{8}$, W can be represented by a single asymptotic expansion in terms of trigonometric functions throughout the interior of the interval $0<Y<1$. However for $c^2<\frac{3}{8}$ there are two turning points at $Y_{\pm}$, so W can be represented by trigonometric functions between 0 and Y_- and between Y_+ and 1, and by exponential functions between Y_- and Y_+. It can be represented by Airy functions in the neighbourhood of each turning point. In addition, in both cases the coefficient of W'' in (2.2) vanishes at $Y=0$ and $Y=1$. Therefore separate boundary-layer expansions, which involve Bessel functions, are needed near these points.

The conclusion is that for $c^2 > \frac{3}{8}$, W can be represented by three asymptotic expansions – one in the interior and two near the boundaries. For $c^2<\frac{3}{8}$, W can be represented by seven asymptotic expansions: two Bessel function expansions near the boundaries, two trigonometric function expansions outside the turning points, two Airy function expansions near the turning points, and one exponential function expansion between the turning points. In each case the various expansions must be matched to the expansions in adjacent intervals. This matching ultimately leads to an eigenvalue equation or dispersion equation relating k and c.

Let us begin with the expansion in the boundary layer near $Y=0$. Since W is finite at $Y=0$, we shall set $W(0)=1$. We introduce the stretched variable $\xi=k^2Y$ and write $W(Y,k) \sim W^0(\xi)$. Upon introducing these variables into (2.2), and retaining terms of the highest order in k, we obtain

$$\xi W^0_{\xi\xi} + W^0_\xi + 2(c+\tfrac{1}{2})^2 W^0 = 0. \tag{3.2}$$

The solution of (3.2) which satisfies $W^0(0) = 1$ is the Bessel function

$$W^0(\xi) = J_0[|c+\tfrac{1}{2}|(8\xi)^{\frac{1}{2}}].$$

Then by setting $\xi = k^2 Y$ in this expression for $W^0(\xi)$ we obtain the result,

$$W(Y,k) \sim J_0[|c+\tfrac{1}{2}|k(8Y)^{\frac{1}{2}}], \quad 0 \leqslant Y \ll 1. \tag{3.3}$$

For $kY^{\frac{1}{2}}$ large, (3.3) yields

$$W(Y,k) \sim [\pi k|c+\tfrac{1}{2}|(2Y)^{\frac{1}{2}}]^{-\frac{1}{2}} \cos[k|c+\tfrac{1}{2}|(8Y)^{\frac{1}{2}} - \tfrac{1}{4}\pi], \quad k^2 Y \gg 1. \tag{3.4}$$

Next we consider the interval $0 < Y < 1$ for $c^2 > \frac{3}{8}$. Since the coefficient of k^2 in (2.2) is positive in this interval, we use the WKB method to obtain an expansion of W. Thus we write

$$W(Y,k) \sim A(Y)\cos[kS(Y)+\theta], \tag{3.5}$$

with θ a constant. Then we substitute (3.5) into (2.2) and equate to zero the coefficients of k^2 and of k to obtain

$$(S')^2 = \frac{2(Y-c-\frac{1}{2})^2}{Y(1-Y)} - 1, \quad A' + \left(\frac{S''}{2S'} + \frac{1-2Y}{2Y(1-Y)}\right)A = 0. \tag{3.6}$$

The solutions of these equations are

$$S(Y) = \int_0^Y \left[\frac{2(y-c-\frac{1}{2})^2}{y(1-y)} - 1\right]^{\frac{1}{2}} \mathrm{d}y, \quad A(Y) = A_0[Y(1-Y)]^{-\frac{1}{2}}\left[\frac{2(Y-c-\frac{1}{2})^2}{Y(1-Y)} - 1\right]^{-\frac{1}{4}}. \tag{3.7}$$

There is no loss of generality in choosing the positive square root for S because the cosine (3.5) is even, nor in setting $S(0) = 0$ because the constant θ is arbitrary.

To determine θ and A_0 we first expand $S(Y)$ and $A(Y)$ for Y near zero to get $S(Y) \sim |c+\frac{1}{2}|(8Y)^{\frac{1}{2}}$ and $A(Y) \sim A_0|c+\frac{1}{2}|^{-\frac{1}{2}}(2Y)^{-\frac{1}{4}}$. Then we use these values in (3.5) to obtain

$$W(Y,k) \sim \frac{A_0}{|c+\frac{1}{2}|^{\frac{1}{2}}(2Y)^{\frac{1}{4}}} \cos[k|c+\tfrac{1}{2}|(8Y)^{\frac{1}{2}} + \theta], \quad 0 < Y \ll 1. \tag{3.8}$$

Now we match (3.8) with (3.4). We see that they coincide if $\theta = -\frac{1}{4}\pi$ and $A_0 = (\pi k)^{-\frac{1}{2}}$. Upon using these values in (3.7), and then using (3.5), we obtain

$$\left.\begin{aligned} &W(Y,k) \sim [\pi k Y(1-Y)]^{-\frac{1}{2}}\left[\frac{2(Y-c-\frac{1}{2})^2}{Y(1-Y)} - 1\right]^{-\frac{1}{4}} \cos[kS(Y) - \tfrac{1}{4}\pi] \\ &0 < Y < 1 \quad \text{if} \quad c^2 > \tfrac{3}{8}; \quad 0 < Y < Y_- \quad \text{if} \quad c^2 < \tfrac{3}{8}. \end{aligned}\right\} \tag{3.9}$$

We have indicated that (3.9) holds also in the interval $0 < Y < Y_-$ when $c^2 < \frac{3}{8}$ since the coefficient of k^2 in (2.2) is positive throughout this interval.

To complete the determination of W when $c^2 > \frac{3}{8}$ we must construct an expansion of W in the boundary layer near $Y = 1$. To do so we introduce the stretched variable $\eta = k^2(1-Y)$ and write $W(Y,k) \sim W^{(1)}(\eta)$. Then by using these variables in (2.2) we find that $W^{(1)}$ satisfies (3.2) with η replacing ξ and $c-\frac{1}{2}$ replacing $c+\frac{1}{2}$, so $W^{(1)}$ is given by $J_0[|c-\frac{1}{2}|(8\eta)^{\frac{1}{2}}]$. Thus we obtain

$$W(Y,k) \sim AJ_0[|c-\tfrac{1}{2}|k[8(1-Y)]^{\frac{1}{2}}], \quad 0 \leqslant 1-Y \ll 1. \tag{3.10}$$

For $k(1-Y)^{\frac{1}{2}}$ large (3.10) becomes

$$W(Y,k) \sim A\{\pi k|c-\tfrac{1}{2}|[2(1-Y)]^{\frac{1}{2}}\}^{-\frac{1}{2}}\cos\{k|c-\tfrac{1}{2}|[8(1-Y)]^{\frac{1}{2}}-\tfrac{1}{4}\pi\}, \quad k^2(1-Y) \gg 1. \tag{3.11}$$

Finally we must match (3.11) with (3.9) by expanding (3.9) for Y near 1. First we expand $S(Y)$ given by (3.7) to obtain $S(Y) \sim S(1)-\sqrt{8}|c-\frac{1}{2}|(1-Y)^{\frac{1}{2}}$ and then we can write (3.9) in the form

$$W(Y,k) \sim [\pi k|c-\tfrac{1}{2}|\sqrt{2}(1-Y)^{\frac{1}{2}}]^{-\frac{1}{2}}\cos[kS(1)-\tfrac{1}{4}\pi-k|c-\tfrac{1}{2}|\sqrt{8}(1-Y)^{\frac{1}{2}}], \quad 1-Y \ll 1. \tag{3.12}$$

Comparison of (3.11) with (3.12) shows that they become identical if $A = (-1)^n$ and $kS(1) = (n+\frac{1}{2})\pi$ where $n \geqslant 0$ is an integer. This last relation determines the eigenvalue k in terms of c, and by using (3.7) we can write it as follows:

$$k = (n+\tfrac{1}{2})\pi \Big/ \int_0^1 \left[\frac{2(y-c-\frac{1}{2})^2}{y(1-y)}-1\right]^{\frac{1}{2}} \mathrm{d}y, \quad |c| > (\tfrac{3}{8})^{\frac{1}{2}}. \tag{3.13}$$

4. Asymptotic solution for $c^2 < \frac{3}{8}$

For $c^2 < \frac{3}{8}$ the boundary-layer expansions (3.3) and (3.10) still apply, and the WKB expansion (3.9) holds for $0 < Y < Y_-$. We shall now construct the additional expansions needed to cover the entire range of Y. First we must expand (3.9) for Y near Y_-. From (3.7) we obtain

$$S(Y) = S(Y_-)-\tfrac{2}{3}\beta_-^{\frac{1}{2}}(Y_--Y)^{\frac{3}{2}}+\dots, \quad \beta_- = \frac{(3-8c^2)^{\frac{1}{2}}}{Y_-(1-Y_-)}. \tag{4.1}$$

Then (3.9) can be written as

$$W(Y,k) \sim [\pi k Y_-(1-Y_-)]^{-\frac{1}{2}}[\beta_-(Y_--Y)]^{-\frac{1}{4}}\cos[kS(Y_-)-\tfrac{1}{4}\pi-\tfrac{2}{3}k\beta_-^{\frac{1}{2}}(Y_--Y)^{\frac{3}{2}}], \quad 0 < Y_--Y \ll 1. \tag{4.2}$$

Now for Y in the boundary layer near Y_- we introduce the stretched variable $x = k^{\frac{2}{3}}(Y-Y_-)$ and write $W(Y,k) \sim W^{(2)}(x)$. Then (2.2) yields

$$W^{(2)}_{xx}-\beta_- x W^{(2)} = 0. \tag{4.3}$$

The solution of (4.3) can be written in terms of Airy functions as

$$c_+ \operatorname{Ai}(\beta_-^{\frac{1}{3}}x)+c_- \operatorname{Bi}(\beta_-^{\frac{1}{3}}x).$$

Thus in the original variables we have

$$W(Y,k) \sim c_+ \operatorname{Ai}[\beta_-^{\frac{1}{3}}k^{\frac{2}{3}}(Y-Y_-)]+c_- \operatorname{Bi}[\beta_-^{\frac{1}{3}}k^{\frac{2}{3}}(Y-Y_-)], \quad |Y-Y_-| \ll 1. \tag{4.4}$$

For $k^{\frac{2}{3}}(Y-Y_-) \ll -1$, (4.4) becomes

$$W(Y,k) \sim \pi^{-\frac{1}{2}}k^{-\frac{1}{6}}\beta_-^{-\frac{1}{12}}(Y_--Y)^{-\frac{1}{4}} \times [c_+ \sin(\tfrac{2}{3}k\beta_-^{\frac{1}{2}}(Y_--Y)^{\frac{3}{2}}+\tfrac{1}{4}\pi)+c_- \cos(\tfrac{2}{3}k\beta_-^{\frac{1}{2}}(Y_--Y)^{\frac{3}{2}}+\tfrac{1}{4}\pi)], \quad k^{\frac{2}{3}}(Y-Y_-) \ll -1. \tag{4.5}$$

Matching of (4.5) with (4.2) yields the following equations for c_+ and c_-:

$$\frac{\sin kS(Y_-)}{[\pi k Y_-(1-Y_-)]^{\frac{1}{2}}\beta_-^{\frac{1}{4}}} = \frac{c_+}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_-^{\frac{1}{12}}}, \qquad \frac{\cos kS(Y_-)}{[\pi k Y_-(1-Y_-)]^{\frac{1}{2}}\beta_-^{\frac{1}{4}}} = \frac{c_-}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_-^{\frac{1}{12}}}. \tag{4.6}$$

Next in the interval $Y_- < Y < Y_+$, where the coefficient of k^2 in (2.2) is negative, we obtain the following exponential form of the WKB expansion:

$$W(Y,k) \sim [\pi k Y(1-Y)]^{-\frac{1}{2}}\left[1-\frac{2(Y-c-\frac{1}{2})^2}{Y(1-Y)}\right]^{-\frac{1}{4}}[\alpha_+ \mathrm{e}^{kR(Y)}+\alpha_- \mathrm{e}^{-kR(Y)}],$$
$$Y_- < Y < Y_+. \quad (4.7)$$

Here $R(Y)$ is defined by

$$R(Y) = \int_{Y_-}^{Y}\left[1-\frac{2(y-c-\frac{1}{2})^2}{y(1-y)}\right]^{\frac{1}{2}}\mathrm{d}y. \quad (4.8)$$

For Y near Y_-, (4.7) becomes

$$W(Y,k) \sim [\pi k Y_-(1-Y_-)]^{-\frac{1}{2}}\beta_-^{-\frac{1}{4}}(Y-Y_-)^{-\frac{1}{4}}$$
$$\times\{\alpha_+ \exp[\tfrac{2}{3}k\beta_-^{\frac{1}{2}}(Y-Y_-)^{\frac{3}{2}}]+\alpha_- \exp[-\tfrac{2}{3}k\beta_-^{\frac{1}{2}}(Y-Y_-)^{\frac{3}{2}}]\}, \quad 0 < Y-Y_- \ll 1. \quad (4.9)$$

Matching (4.9) with (4.7) gives the following expressions for α_+ and α_-:

$$\frac{\alpha_+}{[\pi k Y_-(1-Y_-)]^{\frac{1}{2}}\beta_-^{\frac{1}{4}}} = \frac{c_-}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_-^{\frac{1}{12}}}, \qquad \frac{\alpha_-}{[\pi k Y_-(1-Y_-)]^{\frac{1}{2}}\beta_-^{\frac{1}{4}}} = \frac{c_+}{2\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_-^{\frac{1}{12}}}. \quad (4.10)$$

Upon using (4.6) for c_+ and c_- in (4.11) we get the connection formulae

$$\alpha_- = \tfrac{1}{2}\sin kS(Y_-), \quad \alpha_+ = \cos kS(Y_-). \quad (4.11)$$

This completes the determination of the expansion (4.7).

To expand (4.7) for Y near Y_+ we write

$$R(Y) = R(Y_+) - \tfrac{2}{3}\beta_+^{\frac{1}{2}}(Y_+-Y)^{\frac{3}{2}}+\ldots, \quad \beta_+ = \frac{(3-8c^2)^{\frac{1}{2}}}{Y_+(1-Y_+)}. \quad (4.12)$$

Then (4.7) becomes, for Y near Y_+,

$$W(Y,k) \sim [\pi k Y_+(1-Y_+)]^{-\frac{1}{2}}\beta_+^{-\frac{1}{4}}(Y_+-Y)^{-\frac{1}{4}}\{\cos kS(Y_-)\exp[kR(Y_+)-\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y_+-Y)^{\frac{3}{2}}]$$
$$+\tfrac{1}{2}\sin kS(Y_-)\exp[-kR(Y_+)+\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y_+-Y)^{\frac{3}{2}}]\}, \quad 0 < Y_+-Y \ll 1. \quad (4.13)$$

By proceeding as in the case of the turning point at Y_-, we find that near the turning point Y_+, W satisfies the Airy equation (4.3) with β_- replaced by β_+ and with $x = k^{\frac{2}{3}}(Y_+-Y)$. Thus the expansion of W near Y_+ is given by

$$W(Y,k) \sim d_+ \operatorname{Ai}[\beta_+^{\frac{1}{3}}k^{\frac{2}{3}}(Y_+-Y)]+d_- \operatorname{Bi}[\beta_+^{\frac{1}{3}}k^{\frac{2}{3}}(Y_+-Y)], \quad |Y_+-Y| \ll 1. \quad (4.14)$$

Expanding (4.14) for $k^{\frac{2}{3}}(Y_+-Y) \gg 1$ yields

$$W(Y,k) \sim \pi^{-\frac{1}{2}}k^{-\frac{1}{6}}\beta_+^{-\frac{1}{12}}(Y_+-Y)^{-\frac{1}{4}}$$
$$\times\{\tfrac{1}{2}d_+ \exp[-\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y_+-Y)^{\frac{3}{2}}]+d_- \exp[\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y_+-Y)^{\frac{3}{2}}]\}, \quad k^{\frac{2}{3}}(Y_+-Y) \gg 1. \quad (4.15)$$

Matching (4.15) with (4.13) gives the following equations for d_+ and d_-:

$$\frac{d_+}{2\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_+^{\frac{1}{12}}} = \frac{\cos kS(Y_-)\,\mathrm{e}^{kR(Y_+)}}{[\pi k Y_+(1-Y_+)]^{\frac{1}{2}}\beta_+^{\frac{1}{4}}}, \quad \frac{d_-}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_+^{\frac{1}{12}}} = \frac{\sin kS(Y_-)\,\mathrm{e}^{-kR(Y_+)}}{2[\pi k Y_+(1-Y_+)]^{\frac{1}{2}}\beta_+^{\frac{1}{4}}}. \quad (4.16)$$

Now that the expansion (4.14) is determined, we expand it for $k^{\frac{2}{3}}(Y_+-Y) \ll -1$ to obtain

$$W(Y,k) \sim \pi^{-\frac{1}{2}}k^{-\frac{1}{6}}\beta_+^{-\frac{1}{12}}(Y-Y_+)^{-\frac{1}{4}}$$
$$\times[d_+\sin(\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y-Y_+)^{\frac{3}{2}}+\tfrac{1}{4}\pi)+d_-\cos(\tfrac{2}{3}k\beta_+^{\frac{1}{2}}(Y-Y_+)^{\frac{3}{2}}+\tfrac{1}{4}\pi)], \quad k^{\frac{2}{3}}(Y-Y_+) \gg 1.$$
$$(4.17)$$

Next we consider the interval $Y_+ < Y < 1$ where the coefficient of k^2 is positive. By using the usual WKB method, as in the interval $0 < Y < Y_-$, we obtain

$$W(Y,k) \sim [\pi k Y(1-Y)]^{-\frac{1}{2}}\left[\frac{2(Y-c-\frac{1}{2})^2}{Y(1-Y)}-1\right]^{-\frac{1}{4}}$$
$$\times\{b_+\cos[kT(Y)-\tfrac{1}{4}\pi]+b_-\sin[kT(Y)-\tfrac{1}{4}\pi]\}, \quad Y_+ < Y < 1. \quad (4.18)$$

Here $T(Y)$ is defined by

$$T(Y) = \int_Y^1 \left[\frac{2(y-c-\frac{1}{2})^2}{y(1-y)}-1\right]^{\frac{1}{2}} dy. \quad (4.19)$$

For Y near Y_+, (4.18) becomes

$$W(Y,k) \sim [\pi k Y_+(1-Y_+)]^{-\frac{1}{2}}\beta_+^{-\frac{1}{4}}(Y-Y_+)^{-\frac{1}{4}}\left\{b_+\cos\left[kT(Y_+)-\tfrac{1}{4}\pi-\frac{2k}{3}\beta_+^{\frac{1}{2}}(Y-Y_+)^{\frac{3}{2}}\right]\right.$$
$$\left.+b_-\sin\left[kT(Y_+)-\tfrac{1}{4}\pi-\frac{2k}{3}\beta_+^{\frac{1}{2}}(Y-Y_+)^{\frac{3}{2}}\right]\right\}, \quad 0 < Y-Y_+ \ll 1. \quad (4.20)$$

For Y near 1, (4.18) simplifies to

$$W(Y,k) \sim [\pi k|c-\tfrac{1}{2}|(1-Y)]^{-\frac{1}{2}}2^{-\frac{1}{4}}(1-Y)^{-\frac{1}{4}}\{b_+\cos[k|c-\tfrac{1}{2}|\sqrt{8}(1-Y)^{\frac{1}{2}}-\tfrac{1}{4}\pi]$$
$$-b_-\sin[k|c-\tfrac{1}{2}|\sqrt{8}(1-Y)^{\frac{1}{2}}-\tfrac{1}{4}\pi]\}, \quad 0 < 1-Y \ll 1. \quad (4.21)$$

Now we match (4.21) to the boundary-layer expansion near $Y = 1$, which is given by (3.10) and in expanded form by (3.11). Matching shows that $b_- = 0$ and $b_+ = A$. Then we match (4.20) with (4.17) to get

$$\frac{A\sin kT(Y_+)}{[\pi k Y_+(1-Y_+)]^{\frac{1}{2}}\beta_+^{\frac{1}{4}}} = \frac{d_+}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_+^{\frac{1}{12}}}, \quad \frac{A\cos kT(Y_+)}{[\pi k Y_+(1-Y_+)]^{\frac{1}{2}}\beta_+^{\frac{1}{4}}} = \frac{d_-}{\pi^{\frac{1}{2}}k^{\frac{1}{6}}\beta_+^{\frac{1}{12}}}. \quad (4.22)$$

Using (4.16) to eliminate d_+ and d_- from (4.22), and then eliminating A between the two equations (4.22) yields the dispersion equation

$$\cos kS(Y_-)\cos kT(Y_+) = \tfrac{1}{4}e^{-2kR(Y_+)}\sin kS(Y_-)\sin kT(Y_+), \quad c^2 < \tfrac{3}{8}. \quad (4.23)$$

5. Analysis of the dispersion equation

We have obtained the dispersion equations (4.23) for $c^2 < \frac{3}{8}$ and (3.13) for $c^2 > \frac{3}{8}$. Now we shall analyse those equations, beginning with (3.13) which is

$$k = (n+\tfrac{1}{2})\pi \Big/ \int_0^1 \left[\frac{2(y-c-\frac{1}{2})^2}{y(1-y)}-1\right]^{\frac{1}{2}} dy, \quad c^2 > \tfrac{3}{8}. \quad (5.1)$$

This equation shows that the graph of k versus c consists of an infinite number of branches. They all can be obtained from the lowest branch with $n = 0$. To determine this branch we need merely evaluate the integral in (5.1) as a function of c. For this purpose it is convenient to set $y = \frac{1}{2}(1+\sin\theta)$, and then (5.1) becomes

$$k = (n+\tfrac{1}{2})\pi \Big/ \int_{-\pi/2}^{\pi/2} [2(\tfrac{1}{2}\sin\theta-c)^2+\tfrac{1}{4}\sin^2\theta-\tfrac{1}{4}]^{\frac{1}{2}}d\theta, \quad c^2 > \tfrac{3}{8}. \quad (5.2)$$

For each $n \geqslant 0$ this equation yields a unique real value of k for each $|c| > (\frac{3}{8})^{\frac{1}{2}}$, and the value of k decreases as $|c|$ increases. For $|c| \gg 1$ it yields

$$k = \frac{(n+\frac{1}{2})}{\sqrt{2}|c|} + o\left(\frac{1}{|c|}\right), \quad |c| \gg 1. \tag{5.3}$$

Thus for $|c| > (\frac{3}{8})^{\frac{1}{2}}$, there is no instability.

When $|c| < (\frac{3}{8})^{\frac{1}{2}}$ we must consider (4.23). By neglecting the exponentially small right-hand side we get $\cos kS(Y_-) \cos kT(Y_+) = 0$, which has the two families of solutions

$$k_n = \frac{(n+\frac{1}{2})\pi}{S(Y_-)}, \quad n = 0, 1, 2, \ldots, \tag{5.4}$$

$$k_m = \frac{(m+\frac{1}{2})\pi}{T(Y_+)}, \quad m = 0, 1, 2, \ldots. \tag{5.5}$$

Equations (5.4) and (5.5) yield two sets of curves of k versus c. The curves for $n = 0$ and $m = 0$ determine all the others since $k_n(c) = (2n+1)\,k_0(c)$ in each case. To obtain the two curves for $n = 0$ and $m = 0$ it is necessary to evaluate the integrals $S(Y_-)$ and $T(Y_+)$ as functions of c. However, $T[Y_+(c), c] = S[Y_-(-c), -c]$ in view of the invariance of the problem under the transformation $(c, Y) \rightarrow (-c, 1-Y)$ which was pointed out in §2. Thus it suffices to evaluate just one integral, $S[Y_-(c), c]$, as a function of c.

From (3.1) we find that the turning point $Y_-(c)$ tends to zero as c decreases to $-\frac{1}{2}$ and $Y_+(c)$ tends to one as c increases to $\frac{1}{2}$. Therefore $S[Y_-(c), c] \rightarrow 0$ as c decreases to $-\frac{1}{2}$ and $T[Y_+(c), c] \rightarrow 0$ as c increases to $\frac{1}{2}$. By evaluating the integral $S[Y_-(c), c]$ for c near $-\frac{1}{2}$ we obtain $S[Y_-(c), c] = 4(c+\frac{1}{2})^2 + \ldots$ as c decreases to $-\frac{1}{2}$. Therefore

$$T[Y_+(c), c] = 4(\tfrac{1}{2}-c)^2 + \ldots$$

as c increases to $\frac{1}{2}$. By using these results in (5.4) and (5.5) we get

$$k_n \sim \frac{(n+\frac{1}{2})\pi}{4(c+\frac{1}{2})^2} \quad \text{as} \quad c \searrow -\tfrac{1}{2}, \quad n = 0, 1, 2, \ldots, \tag{5.6}$$

$$k_m \sim \frac{(m+\frac{1}{2})\pi}{4(\frac{1}{2}-c)^2} \quad \text{as} \quad c \nearrow +\tfrac{1}{2}, \quad m = 0, 1, 2, \ldots. \tag{5.7}$$

Thus all the curves of the first family tend to infinity as c decreases to $-\frac{1}{2}$ while those of the second family tend to infinity as c increases to $\frac{1}{2}$. When c is plotted versus k, the curves have asymptotes at $c = -\frac{1}{2}$ or $c = +\frac{1}{2}$ as $k \rightarrow \infty$.

This result and (5.3) show that each curve has two asymptotes, and therefore has the general form of an hyperbola. On one set of curves, k increases from zero at $c = +\infty$ to $+\infty$ at $c = -\frac{1}{2}$. The other set is the image of this set in the k-axis. Therefore each curve of one set crosses every curve of the other set at some value of c in the interval $-\frac{1}{2} < c < \frac{1}{2}$. At these intersections (5.4) and (5.5) yield the same value of k. But a boundary-value problem for a second-order equation cannot have a double eigenvalue. Therefore we must take account of the small right-hand side of (4.23) in solving for k at and near the intersections.

To do so we denote by c_{nm} the value of c at which (5.4) and (5.5) yield the same value of $k = k_{nm}$ for given n and m:

$$\frac{(n+\frac{1}{2})\pi}{S(Y_-, c_{nm})} = \frac{(m+\frac{1}{2})\pi}{T(Y_+, c_{nm})} = k_{nm}. \tag{5.8}$$

Then we introduce ϵ and δ defined by

$$k-k_{nm}=\frac{\epsilon}{S(Y_-,c_{nm})},\quad c-c_{nm}=\delta. \tag{5.9}$$

Upon using (5.8) and (5.9) in (4.23) and expanding the trigonometric functions for ϵ and δ small, we can write (4.23) in the form

$$\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\left[\epsilon+(n+\tfrac{1}{2})\,\pi\frac{S_c(Y_-,c_{nm})}{S(Y_-,c_{nm})}\delta\right]\left[\epsilon+(n+\tfrac{1}{2})\,\pi\frac{T_c(Y_+,c_{nm})}{T(Y_+,c_{nm})}\delta\right]+\dots$$
$$=\tfrac{1}{4}\exp\left[-2(n+\tfrac{1}{2})\,\pi\frac{R(Y_+,c_{nm})}{S(Y_-,c_{nm})}\right]+\dots. \tag{5.10}$$

Here $S_c=\partial S/\partial c$ and $T_c=\partial T/\partial c$.

When the higher-order terms are omitted, (5.10) becomes a quadratic equation for δ as a function of ϵ. The solution can be written in the following form in which (5.9) is used to replace δ by $c-c_{nm}$:

$$c-c_{nm}=-\left[\frac{S(Y_-,c_{nm})}{S_c(Y_-,c_{nm})}+\frac{T(Y_+,c_{nm})}{T_c(Y_+,c_{nm})}\right]\frac{\epsilon}{2(n+\frac{1}{2})\,\pi}$$
$$\pm\frac{1}{2(n+\frac{1}{2})\,\pi}\left\{\left(\frac{S}{S_c}-\frac{T}{T_c}\right)^2\epsilon^2+\frac{ST}{S_cT_c}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)\exp\left[-\frac{2(n+\frac{1}{2})\,\pi R(Y_+,c_{nm})}{S(Y_-,c_{nm})}\right]\right\}^{\frac{1}{2}}. \tag{5.11}$$

The arguments of S, T, S_c and T_c in the square root are the same as those in the preceding term.

From the definition (3.7) of S we see that $S>0$ and $S_c>0$ for $0<Y\leqslant Y_-$ and all real c. From (4.19) we see that $T>0$ for $Y_+\leqslant Y<1$ and $T_c(Y_+,c)=-S_c(Y_-,-c)$. Therefore the coefficient of the exponential in (5.11) is negative. Consequently the graph of δ versus ϵ, or of $c-c_{nm}$ versus $k-k_{nm}$, is an hyperbola. There is an interval of ϵ or of $k-k_{nm}$ within which $c-c_{nm}$ is complex. That interval lies between the two points at which the square root in (5.11) vanishes. These points are given by

$$k-k_{nm}=\frac{\pm 1}{S(Y_-,c_{nm})}\left(\left|\frac{S}{S_c}-\frac{T}{T_c}\right|\right)^{-1}\left[-\frac{ST}{S_cT_c}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)\right]^{\frac{1}{2}}\exp\left[-\frac{(n+\frac{1}{2})\,\pi R(Y_+)}{S(Y_-)}\right]. \tag{5.12}$$

Between these points the imaginary part of c, c_{I}, is given by the square-root term in (5.11). Since there is one root with a positive value of c_{I}, the basic flow is unstable in this interval of k. Both $k-k_{nm}$ and c_{I} decreases exponentially with n. Therefore the growth rate of the unstable mode, and the interval within which it occurs, are both very small for large or moderate values of n.

From (3.7) and (4.19) we see that $S(Y_-,0)=T(Y_+,0)$. Therefore (5.6) can be satisfied by setting $m=n$ and $c_{nm}=0$. We also see that $T_c(Y_+,0)=-S_c(Y_-,0)$ so the first term in (5.11) vanishes and the second term simplifies to yield

$$c=\pm\frac{S(Y_-,0)}{(n+\frac{1}{2})\,\pi S_c(Y_-,0)}\left\{\epsilon^2-\tfrac{1}{4}\exp\left[-\frac{2(n+\frac{1}{2})\,\pi R(Y_+,0)}{S(Y_-,0)}\right]\right\}^{\frac{1}{2}}, \tag{5.13}$$
$$\epsilon=S(Y_-,0)\,k-(n+\tfrac{1}{2})\,\pi.$$

The endpoints of the unstable interval are, from (5.12) or (5.13),

$$k-\frac{(n+\frac{1}{2})\,\pi}{S(Y_-,0)}=\frac{\pm 1}{S(Y_-,0)}\exp\left[-\frac{(n+\frac{1}{2})\,\pi R(Y_+,0)}{S(Y_-,0)}\right]. \tag{5.14}$$

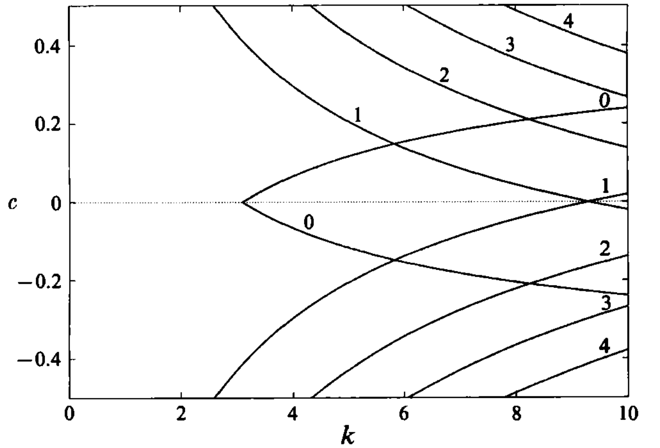

FIGURE 1. Dispersion curves showing the real part of the dimensionless phase speed c as a function of the dimensionless wavenumber k. The curves are based upon (5.4) and (5.5), which are obtained by asymptotic solution of (2.2) and (2.3) for $k \gg 1$. Surrounding each point of intersection is an interval of k within which c is complex. These instability intervals are given by (5.12), except for the first one. The value of n is shown on each curve.

By using the preceding results we can now draw the curves of c versus k by proceeding as follows.:

(*a*) We evaluate numerically the integral in (5.2) for a set of values of $c > (\frac{3}{8})^{\frac{1}{2}}$, and then calculate k from (5.2) for $n = 0$, or any other value. For c large we can use (5.3).

(*b*) We evaluate numerically $S[Y_-(c), c)$ given by (3.7) with $Y_-(c)$ given by (3.1), for a set of values of c in the interval $(\frac{3}{8})^{\frac{1}{2}} > c > -\frac{1}{2}$. Then we calculate k from (5.4) for $n = 0$, or any other value. For c near $-\frac{1}{2}$ we can use (5.6).

(*c*) The second family of curves is the image of the first family in the k-axis.

(*d*) At and near the points of intersection of curves of the two families we replace the intersecting curves by the hyperbola (5.11). In the gap (5.12) c is complex and it is given by (5.11).

Figure 1 gives the curves $k_n(c)$ for c in the range $-\frac{1}{2} < c < \frac{1}{2}$, and k_n in the range $0 < k_n < 10$. They are obtained by the procedures (*b*) and (*c*) above. For $n = 0$ we show only the parts of the curves to the right of their point of intersection on the axis $c = 0$. The curves in figure 1 are in good agreement with those given by Hayashi & Young (1987) in their figure 2. Both of these figures apply when their parameter ϕ is small. Near the points of intersection of the curves in figure 1 there are small intervals of instability given by (5.12). Within these intervals c is given by the complex expression (5.11). These results simplify to (5.14) and (5.13) for the instability intervals on the axis $c = 0$. The first of these instability intervals contains the intersection point of the curves with $n = 0$. The left endpoint of this interval is not given correctly by (5.14) with the minus sign and $n = 0$. This is because the interval extends to $k = 0$, whereas our expansion is valid only for k large. It is remarkable that the curves are given so well for the values of k in the figure, which are not large, in agreement with the usual behaviour of asymptotic expansions.

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